Senior Mathematical Challenge

Thursday 5th November 2015

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some problems for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with all steps explained, and not based on the assumption that one of the given alternatives is correct. In some cases we have added a commentary to indicate the sort of thinking that led to our solution. You should not include commentary of this kind in your written solutions, but we hope that these solutions, without the commentary, provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	A	B	B	A	E	B	D	B	D	C	A	E	E	C	A	C	D	A	E	C	B	D	C	C

1. What is $2015^2 - 2016 \times 2014$?

A -2015 B -1 C 0 D 1 E 2015

Solution **D**

Commentary

This question could be answered by doing two multiplications, to evaluate 2015^2 and 2016×2014, and then one subtraction. It should be clear that this method cannot be what was intended as it would be tedious and take up a lot of time. There must be a better method. The clue is that the product 2016×2014 may be rewritten as $(2015 + 1) \times (2015 - 1)$, and that this last expression is equivalent to the difference of two squares, namely, $2015^2 - 1^2$. This leads to the solution given below.

[See Question 22 for another case where the factorization of the difference of two squares is useful.]

We have

$$\begin{aligned} 2015^2 - 2016 \times 2014 &= 2015^2 - (2015 + 1) \times (2015 - 1) \\ &= 2015^2 - (2015^2 - 1) \\ &= 2015^2 - 2015^2 + 1 \\ &= 1. \end{aligned}$$

2. What is the sum of all the solutions of the equation $6x = \dfrac{150}{x}$?

A 0 B 5 C 6 D 25 E 156

Solution **A**

We have,

$$\begin{aligned} 6x = \frac{150}{x} &\Leftrightarrow 6x^2 = 150 \\ &\Leftrightarrow x^2 = 25 \\ &\Leftrightarrow x = -5 \text{ or } x = 5. \end{aligned}$$

Therefore the equation has the two solutions -5 and 5. It follows that the sum of all the solutions of the equation is $-5 + 5 = 0$.

Note that here the symbol $\Leftrightarrow$ stands for *if, and only if*.

3. When Louise had her first car, 50 litres of petrol cost £40. When she filled up the other day, she noticed that 40 litres of petrol cost £50.

By approximately what percentage has the cost of petrol increased over this time?

A 50% B 56% C 67% D 75% E 80%

Solution **B**

We note first that £40 is 40×100 pence, that is, 4000 pence. Therefore when Louise had her first car, the cost of petrol, in pence per litre, was

$$\frac{4000}{50} = 80.$$

Similarly, when she filled up the other day, the cost of petrol, in pence per litre, was

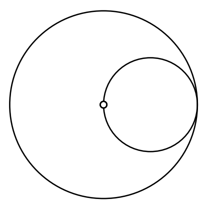
$$\frac{5000}{40} = 125.$$

It follows that the cost of petrol has increased by $(125 - 80)$ pence per litre, that is, by 45 pence per litre. As a percentage of the original price this increase is

$$\begin{aligned}\frac{45}{80} \times 100 &= \frac{4500}{80}\\ &= \frac{450}{8}\\ &= 56\tfrac{1}{4}.\end{aligned}$$

Therefore the increase in the cost of petrol over the given period is approximately 56%.

4. In the diagram, the smaller circle touches the larger circle and also passes through its centre.



What fraction of the area of the larger circle is outside the smaller circle?

A $\frac{2}{3}$ B $\frac{3}{4}$ C $\frac{4}{5}$ D $\frac{5}{6}$ E $\frac{6}{7}$

Solution **B**

Because the smaller circle touches the larger circle and passes through its centre, the diameter of the smaller circle is half that of the larger circle. It follows that the radius of the smaller circle is half that of the larger circle.

The area of a circle with radius r is πr^2. Therefore, the area of a circle with radius $\frac{1}{2}r$ is $\pi(\frac{1}{2}r)^2 = \frac{1}{4}\pi r^2$. It follows that the area of the smaller circle is $\frac{1}{4}$ of the area of the larger circle. Therefore $\frac{1}{4}$ of the area of the larger circle is inside the smaller circle. Hence $\frac{3}{4}$ of the area of the larger circle is outside the smaller circle.

5. The integer n is the mean of the three numbers 17, 23 and $2n$.

What is the sum of the digits of n?

A 4 B 5 C 6 D 7 E 8

Solution **A**

Because the mean of 17, 23 and $2n$ is n,

$$\frac{17 + 23 + 2n}{3} = n,$$

and therefore,

$$17 + 23 + 2n = 3n,$$

that is,

$$40 + 2n = 3n,$$

from which it follows that

$$n = 40.$$

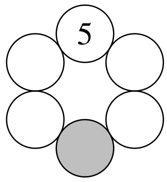
Therefore the sum of the digits of n is $4 + 0 = 4$.

For investigation

5.1 The integer n is the mean of the four numbers 11, 13, 19 and $3n$. What is the value of n?

5.2 Let k be a positive integer, with $k \geq 2$. Show that if n is the mean of the k numbers, $a_1, a_2, \ldots, a_{k-1}$ and $(k-1)n$, then $n = a_1 + a_2 + \cdots + a_{k-1}$.

6. The numbers 5, 6, 7, 8, 9, 10 are to be placed, one in each of the circles in the diagram, so that the sum of the numbers in each pair of touching circles is a prime number. The number 5 is placed in the top circle.



Which number is placed in the shaded circle?

A 6 B 7 C 8 D 9 E 10

Solution **E**

The primes that are the sum of two numbers chosen from 5, 6, 7, 8, 9 and 10 are all greater than 2, and therefore must be odd primes. So if they are the sum of two integers, one of these must be odd and the other even.

It follows that of the two numbers in any pair of touching circles, one must be odd and one must be even. So the numbers must be arranged in the circles so that they are alternately odd and even. Therefore the number in the shaded circle must be even, that is, 6, 8 or 10.

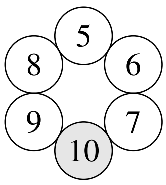
We also deduce from this that the only possible positions for the even numbers not in the shaded circle are the circles adjacent to the top circle which contains the number 5.

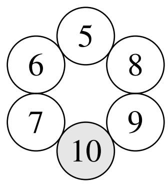
Now the number 10 cannot be adjacent to the top circle because $5 + 10 = 15$ which is not a prime. Therefore 10 must be the number in the bottom circle.

Commentary

In the context of the SMC it is adequate to stop here. For a full solution we need to show that it is possible to put the numbers 6, 7, 8, 9 and 10 in the circles in a way that meets the requirement that the sum of the numbers in each pair of touching circles is prime.

To complete the solution we show that there actually is an arrangement with the number 10 in the bottom circle. It is easy to see that the two arrangements shown in the figures below both meet the requirement that the sum of the numbers in each pair of touching circles is a prime. Note that the only difference between them is that in the figure on the left the numbers 5, 6, 7, 10, 9, 8 go round clockwise, whereas in the figure on the right they go round anticlockwise.





For investigation

6.1 Show that the two arrangements of the numbers given in the above solution are the only arrangements that meet the requirements of the question.

7. Which of the following has the largest value?

A $\dfrac{\left(\frac{1}{2}\right)}{\left(\frac{3}{4}\right)}$ B $\dfrac{1}{\left(\dfrac{\left(\frac{2}{3}\right)}{4}\right)}$ C $\dfrac{\left(\dfrac{\left(\frac{1}{2}\right)}{3}\right)}{4}$ D $\dfrac{1}{\left(\dfrac{2}{\left(\frac{3}{4}\right)}\right)}$ E $\dfrac{\left(\dfrac{1}{\left(\frac{2}{3}\right)}\right)}{4}$

SOLUTION **B**

COMMENTARY

The natural method here is to simplify each fraction to the form $\frac{p}{q}$, where p and q are positive integers, and then to look and see which of these simplified fractions has the largest value.

However, if you adopt this method, you will see that just one of the fractions is greater than 1.

Now, when x and y are positive numbers,

$$\frac{x}{y} > 1 \text{ if, and only if } x > y.$$

It follows that we can answer this question by evaluating the numerator x and the denominator y for each option in turn, and showing that in only one case we have $x > y$. This saves a little work.

In option A the numerator $\frac{1}{2}$ is smaller than the denominator $\frac{3}{4}$ and so the value of the fraction is less than 1.

In option B the numerator is 1 and the denominator is

$$\frac{\left(\frac{2}{3}\right)}{4} = \frac{2}{3} \times \frac{1}{4} = \frac{1}{6}.$$

Since the numerator is greater than the denominator, the value of this fraction is greater than 1.

In option C the numerator is

$$\frac{\left(\frac{1}{2}\right)}{3} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

and the denominator is 4. Since the numerator is less than the denominator, the value of this fraction is less than 1.

In option D the numerator is 1 and the denominator is

$$\frac{2}{\left(\frac{3}{4}\right)} = 2 \div \frac{3}{4} = 2 \times \frac{4}{3} = \frac{8}{3}.$$

Since the numerator is less than the denominator, the value of this fraction is less than 1.

Finally, in option E the numerator is

$$\frac{1}{\left(\frac{2}{3}\right)} = 1 \div \frac{2}{3} = 1 \times \frac{3}{2} = \frac{3}{2}$$

and the denominator is 4. Again, the numerator is less than the denominator, and so the value of this fraction is also less than 1.

It follows that option B has the largest value.

FOR INVESTIGATION

7.1 Arrange the fractions given by the options in Question 7 in order of size.

7.2 The fractions given as options in Question 7 are all built up using 1, 2, 3 and 4 as single digits. There are other possibilities.

Which fraction built up using 1, 2, 3 and 4 as single digits has the largest value?

[The phrase "as single digits" means that the digits cannot be placed next to each other, so that, for example,

$$\frac{432}{1}$$

is not allowed.]

7.3 Which fraction built up using 1, 2, 3, 4 and 5 as single digits has the largest value?

7.4 Simplify the fraction

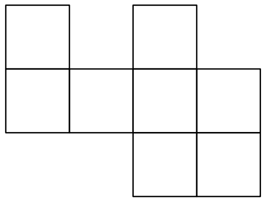
$$\frac{\left(\frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{4}\right)}\right)}{\left(\frac{\left(\frac{5}{6}\right)}{\left(\frac{7}{8}\right)}\right)}$$

by writing it in the form $\frac{p}{q}$, where p and q are positive integers which have no common factor other than 1.

8. The diagram shows eight small squares. Six of these squares are to be shaded so that the shaded squares form the net of a cube.

In how many different ways can this be done?

A 10 B 8 C 7 D 6 E 4



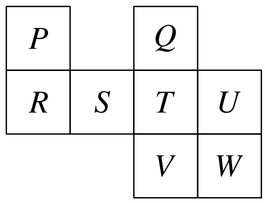
SOLUTION **D**

COMMENTARY

Answering this question as we do to comes down to checking a list of grids obtained from the given diagram by ignoring two of the squares to see in how many cases they form the net of a cube. There are 28 different ways to delete two of the squares in the diagram. Rather a lot to check! So we begin by thinking of a way to cut down the number of grids that we need to check.

When it comes to checking grids, under the conditions of the SMC you will probably have to use your visual imagination to decide whether they could be folded to make cubes. Outside the SMC you could do a practical experiment.

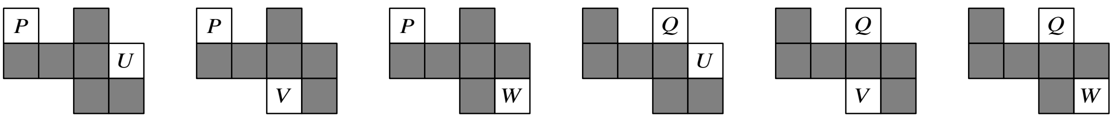
We label the squares as shown in the figure.



We note first that the net of a cube cannot use both the squares P and Q as these must fold to make the same face of any cube.

Each net includes the square T, as otherwise it would not be a connected set of squares. However, no net of a cube can use all four of the squares T, U, V and W. Therefore each net must leave out one of the squares U, V, W.

It follows that there are only six ways in which we might produce the net of the cube by not using a pair of squares. These involve not using one the following pairs of squares: P and U, P and V, P and W, Q and U, Q and V, Q and W. The six grids obtained by ignoring each of these pairs of squares in turn are shown in the figure below.



You should be able to see that each of these is the net of a cube. Therefore there are six different ways in which six of the squares can be used to form the net of a cube.

FOR INVESTIGATION

8.1 Are there any other nets of cubes other than those shown in the figure above?

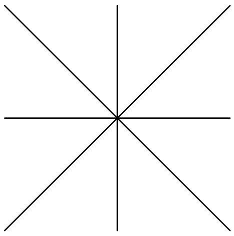
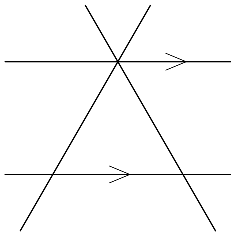
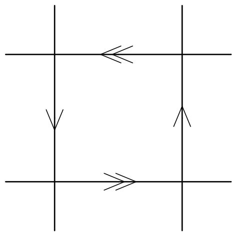
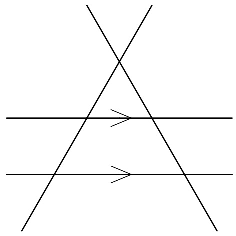
9. Four different straight lines are drawn on a flat piece of paper. The number of points where two or more lines intersect is counted.

Which of the following could *not* be the number of such points?

A 1 B 2 C 3 D 4 E 5

Solution **B**

Each of the figures below shows four straight lines drawn on a flat piece of paper with 1, 3, 4 and 5 points, respectively, where two or more lines intersect.

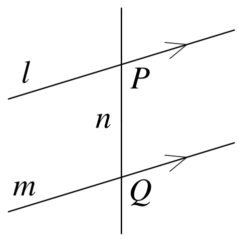
   

It follows that these numbers of intersections are possible. We deduce from this that the correct option is B.

> Commentary
>
> In the context of the SMC, when four of the given options have been eliminated, it is safe to conclude that the remaining option is correct. However, for a complete solution we need to give an argument to show that it is not possible to draw four straight lines so that there are exactly two points of intersection. We now give an argument to prove this.

We show that the attempt to draw four straight lines on a flat piece of paper so that there are just two intersection points is bound to fail.

Suppose that we aim to draw four straight lines so that there are only two points of intersection. Let these points be P and Q.



For P to be a point of intersection, there must be at least two lines through P. Therefore we need to draw at least one line, say l, through P other than the line through P and Q.

Similarly we need to draw at least one line, say m, through Q other than the line through P and Q.

The lines l and m must be parallel since otherwise there would be a third point of intersection.

We can now add the line through P and Q, say n, without creating another intersection point. However, any other line through P would not be parallel to m and hence would create a third intersection point. So we cannot draw such a line. Similarly we cannot draw another line through Q without creating a third intersection point.

So, having drawn the lines l and m, the only line we can add without creating a third intersection point is n. It follows that we cannot draw four straight lines in such a way that there are exactly two intersection points.

FOR INVESTIGATION

9.1 Show that four straight lines can be drawn on a flat piece of paper with exactly six points of intersection.

9.2 Show that it is not possible to draw four straight lines on a flat piece of paper with more than six intersection points.

9.3 What are the possibilities for the number of intersection points when five straight lines are drawn on a flat piece of paper?

9.4 For $n \geq 6$, what are the possibilities for the number of intersection points when n straight lines are drawn on a flat piece of paper?

10. The positive integer n is between 1 and 20. Milly adds up all the integers from 1 to n inclusive. Billy adds up all the integers from $n + 1$ to 20 inclusive. Their totals are the same.

What is the value of n?

A 11 B 12 C 13 D 14 E 15

Solution **D**

Commentary

In the solution below we make use of the formula

$$1 + 2 + \cdots + n = \tfrac{1}{2}n(n + 1)$$

for the sum of the positive integers from 1 to n inclusive.

If you are not familiar with this formula, see Problems 10.3 and 10.4 below.

Because the total that Milly gets is the same as the total that Billy gets, each of their totals is half the sum of the positive integers from 1 to 20 inclusive. We can obtain this total by putting $n = 20$ in the formula $\frac{1}{2}n(n + 1)$. This gives $\frac{1}{2} \times 20 \times 21 = 210$. Therefore the total of the numbers that Milly adds up is half of 210, that is, 105. So we need to find the positive integer n such that $1 + 2 + 3 + \cdots + n = 105$.

Method 1

We find the value of n by trying each of the given options in turn until we find the correct value.

To test whether option A is correct, we need to see if the sum of the integers from 1 to 11 inclusive is equal to 105. By putting $n = 11$ in the formula $\frac{1}{2}n(n + 1)$, we see that

$$1 + 2 + 3 + \cdots + 11 = \tfrac{1}{2} \times 11 \times 12 = 66.$$

So option A is not the correct answer.

We now test the other options in turn. We have

$$1 + 2 + \cdots + 12 = (1 + 2 + \cdots + 11) + 12 = 66 + 12 = 78.$$
$$1 + 2 + \cdots + 13 = (1 + 2 + \cdots + 12) + 13 = 78 + 13 = 91.$$
$$1 + 2 + \cdots + 14 = (1 + 2 + \cdots + 13) + 14 = 91 + 14 = 105.$$

Therefore $n = 14$. So option D is correct.

In the context of the SMC we can stop here.

However, a complete solution should explain why there is no other value of n that works. See Problem 10.1, below.

Method 2

To find the smallest positive integer n such that the sum of the integers from 1 to n, inclusive, is 105, we solve the equation $\frac{1}{2}n(n+1) = 105$. We have

$$\begin{aligned}\tfrac{1}{2}n(n+1) = 105 &\Leftrightarrow n(n+1) = 210\\ &\Leftrightarrow n^2 + n = 210\\ &\Leftrightarrow n^2 + n - 210 = 0\\ &\Leftrightarrow (n+15)(n-14) = 0\\ &\Leftrightarrow n = -15 \text{ or } n = 14.\end{aligned}$$

Because n is a positive integer, we deduce that $n = 14$.

Note that, as in the solution to Question 2, here the symbol $\Leftrightarrow$ stands for *if, and only if*.

For investigation

10.1 Prove that there is exactly one positive integer n that satisfies the equation

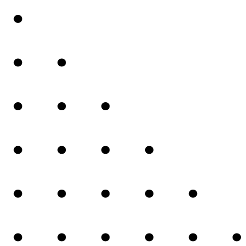
$$\tfrac{1}{2}n(n+1) = 105.$$

10.2 Show that for each positive integer m the equation

$$\tfrac{1}{2}n(n+1) = m$$

has at most one positive integer solution.

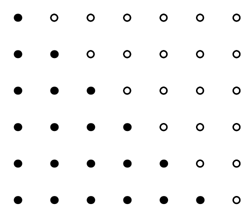
10.3 The numbers obtained by adding up all the positive integers from 1 to n inclusive are called the *triangular numbers*. This is because they correspond to the number of dots in a triangular array. For example, the figure on the right shows an array of dots corresponding to the sum $1 + 2 + 3 + 4 + 5 + 6$.



The notation T_n is often used for the n-th triangular number. That is,

$$T_n = 1 + 2 + 3 + \cdots + n.$$

In the figure on the right we have put together two triangular arrays of dots corresponding to the sum $1 + 2 + 3 + 4 + 5 + 6$. The two triangular arrays together form a rectangle with 6 rows and 7 columns. The rectangle therefore contains 6×7 dots. So the figure illustrates the fact that $2T_6 = 6 \times 7$, and hence that $T_6 = \frac{1}{2}(6 \times 7)$.



Generalize the above to give a proof of the formula $T_n = \frac{1}{2}n(n+1)$. for all positive integers n.

10.4 We can express the sum of the first n positive integers more succinctly using the *sigma* notation. That is, we write $\sum_{k=1}^{n} k$ for the sum $1 + 2 + 3 + \cdots + n$, and, more generally, $\sum_{k=1}^{n} s_k$ for the sum $s_1 + s_2 + s_3 + \cdots + s_n$. We use this notation to give an algebraic proof of the formula for the triangular numbers.

We begin with the observation that

$$(k+1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1.$$

Therefore summing for k going from 1 to n, we obtain

$$\begin{aligned}\sum_{k=1}^{n}\left((k+1)^2 - k^2\right) &= \sum_{k=1}^{n}(2k+1)\\ &= 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.\end{aligned}$$

That is,

$$\sum_{k=1}^{n}\left((k+1)^2 - k^2\right) = 2T_n + n.$$

Show how the left hand side of the above equation simplifies. Then rearrange the equation to obtain the formula for T_n.

10.5 The solution to Question 10 shows that the equation $T_m = \frac{1}{2}T_n$ has the positive integer solution $m = 14$, $n = 20$.

Show that if $m = a$, $n = b$ is a solution of the equation $T_m = \frac{1}{2}T_n$, then so also is $m = 3a + 2b + 2$ and $n = 4a + 3b + 3$.

Deduce that the equation $T_m = \frac{1}{2}T_n$, has infinitely many positive integer solutions. [Note that another way of putting this is to say that there are infinitely many positive integers T such that both T and $2T$ are triangular numbers.]

11. Rahid has a large number of cubic building blocks. Each block has sides of length 4 cm, 6 cm or 10 cm. Rahid makes little towers built from three blocks stacked on top of each other.

How many different heights of tower can he make?

A 6 B 8 C 9 D 12 E 27

SOLUTION **C**

> COMMENTARY
>
> The most straightforward approach here is to list all the possible ways of choosing three of the blocks. For a full solution it is important to do this in a systematic way that makes it clear that every possible case occurs in the list, and no case is listed more than once. We have set out the table in the solution below in a way that we hope makes it clear that we have achieved this.

In the table below we list all combinations of three blocks each of side length 4 cm, 6 cm or 10 cm. In the last column we have give the height of the tower that the given blocks make.

number of 4 cm blocks	number of 6 cm blocks	number of 10 cm blocks	height of tower
3	0	0	12 cm
0	3	0	18 cm
0	0	3	30 cm
2	1	0	14 cm
2	0	1	18 cm
1	2	0	16 cm
0	2	1	22 cm
1	0	2	24 cm
0	1	2	26 cm
1	1	1	20 cm

We see that there are ten ways in which Rahid can choose three blocks with which to make a little tower. However, two of them make towers of the same height, namely 18 cm. So there are 9 different heights of tower that Rahid can make.

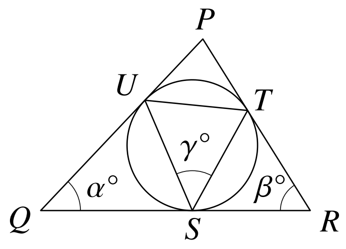
FOR INVESTIGATION

11.1 How many different heights of tower can be built using four of the blocks?

12. A circle touches the sides of triangle PQR at the points S, T and U as shown. Also $\angle PQR = \alpha°$, $\angle PRQ = \beta°$ and $\angle TSU = \gamma°$.

Which of the following gives γ in terms of α and β?

A $\frac{1}{2}(\alpha + \beta)$ B $180 - \frac{1}{2}(\alpha + \beta)$
C $180 - (\alpha + \beta)$ D $\alpha + \beta$
E $\frac{1}{3}(\alpha + \beta)$



Solution **A**

Method 1

From the given fact that the circle touches the sides of the triangle, we deduce that the lines PQ, QR and RP are tangents to the circle. The two tangents from a given point to a circle are of equal length. In particular, $QS = QU$. Therefore the triangle QSU is isosceles, and hence $\angle QSU = \angle QUS$.

Because the sum of the angles in a triangle is 180°, from the triangle QSU we deduce that $\alpha° + \angle QSU + \angle QUS = 180°$.

Since $\angle QSU = \angle QUS$, it follows that $\alpha° + 2\angle QSU = 180°$. We can rearrange this equation to give

$$\angle QSU = \tfrac{1}{2}(180 - \alpha)°.$$

In a similar way, we have that the tangents RS and RT are of equal length. Hence the triangle RTS is isosceles and therefore

$$\angle RST = \tfrac{1}{2}(180 - \beta)°.$$

Because QSR is a straight line, the angles at S on this line have sum 180°, that is,

$$\angle QSU + \angle RST + \gamma° = 180°.$$

Substituting in this equation the expressions for $\angle QSU$ and $\angle RST$ we have already found, we deduce that

$$\tfrac{1}{2}(180 - \alpha)° + \tfrac{1}{2}(180 - \beta)° + \gamma° = 180°.$$

We can rearrange this equation as

$$180° - \tfrac{1}{2}(\alpha + \beta)° + \gamma° = 180°,$$

from which it follows that

$$\gamma = \tfrac{1}{2}(\alpha + \beta).$$

Method 2

As in method 1, PQ and PT are tangents to the circle.

By the *Alternate Angle Theorem*, $\angle PUT = \angle PTU = \gamma°$. Therefore because the angles in the triangle PUT have sum 180°, we have $\angle UPT + \gamma° + \gamma° = 180°$, and therefore

$$\angle UPT = 180° - 2\gamma°.$$

Therefore, as the angles in the triangle PQR have sum 180°,

$$\alpha° + \beta° + (180° - 2\gamma°) = 180°.$$

We can rearrange this last equation to give

$$2\gamma° = \alpha° + \beta°,$$

and it follows that

$$\gamma = \tfrac{1}{2}(\alpha + \beta).$$

FOR INVESTIGATION

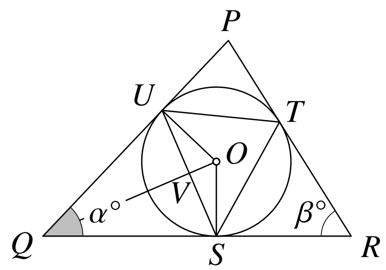
12.1 In Method 1 we used the fact that *the two tangents from a given point to a circle are of equal length.* Find a proof of this (that is, either devise your own proof, look in a book or on the internet, or ask your teacher).

12.2 In Method 2 we have used the *Alternate Segment Theorem* which says that the angle between a tangent to a circle and a chord is equal to the angle subtended by the chord at the circumference of the circle. Find a proof of this theorem.

12.3 Here we consider a third method.

We let O be the centre of the circle. In the figure we have added the lines joining O to the points Q, S, and U. We let V be the point where QO meets US.

Now proceed as follows.



(i) Prove that the triangles QOS and QOU are congruent.
(ii) Deduce that $\angle VQS = \angle VQU = \frac{1}{2}\alpha°$.
(iii) Prove that the triangles QVS and QVU are congruent.
(iv) Deduce that $\angle QVS = 90°$.
(v) Deduce that $\angle USO = \frac{1}{2}\alpha°$.
(vi) Show that, similarly, $\angle TSO = \frac{1}{2}\beta°$.
(vii) Conclude that $\gamma = \frac{1}{2}(\alpha + \beta)$.

13. The Knave of Hearts tells only the truth on Mondays, Tuesdays, Wednesdays and Thursdays. He tells only lies on all the other days. The Knave of Diamonds tells only the truth on Fridays, Saturdays, Sundays and Mondays. He tells only lies on all the other days. On one day last week, they both said, "Yesterday I told lies."

On which day of the week was that?

A Sunday B Monday C Tuesday D Thursday E Friday

SOLUTION **E**

A day on which one of the Knaves says "Yesterday I told lies." is either a day when he is telling the truth but the previous day he was lying, or else a day on which he is telling lies and the previous day he was telling the truth.

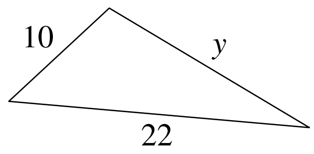
So for the Knave of Hearts that day would have to be a Monday or a Friday. Likewise, for the Knave of Diamonds that day would have to be a Friday or a Tuesday.

Therefore a day on which they both said "Yesterday I told lies." is a Friday.

14. The triangle shown has an area of 88 square units.

What is the value of y?

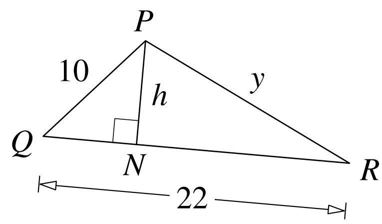
A 17.6 B $2\sqrt{46}$ C $6\sqrt{10}$ D $13\sqrt{2}$ E $8\sqrt{5}$



SOLUTION **E**

METHOD 1

We label the vertices of the triangle as shown in the figure. We let N be the point where the perpendicular from P to QR meets QR, and we let h be the length of PN.



The triangle has area 88. So, from the formula area $= \frac{1}{2}(\text{base} \times \text{height})$ for a triangle, we have

$$88 = \tfrac{1}{2}(22 \times h),$$

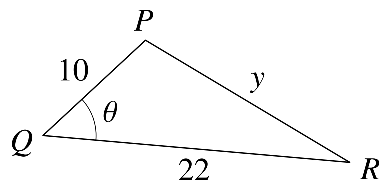
that is, $88 = 11h$. It follows that $h = 8$.

By Pythagoras' Theorem applied to the right-angled triangle QNP, we have $QN^2 + h^2 = 10^2$. Therefore $QN^2 = 10^2 - h^2 = 10^2 - 8^2 = 100 - 64 = 36$. Since QN is a length and therefore positive, we deduce that $QN = 6$. [Alternatively, you could just note that PQN is a right angled triangle with sides in the ratio 5 : 4 : 3.]

It follows that $NR = QR - QN = 22 - 6 = 16$. Now, applying Pythagoras' Theorem to the right-angled triangle PNR, we have $NR^2 + h^2 = y^2$, that is, $16^2 + 8^2 = y^2$. Hence $y^2 = 256 + 64 = 320$. We deduce that $y = \sqrt{320} = \sqrt{64 \times 5} = 8\sqrt{5}$.

METHOD 2

We use the fact that the area of a triangle is $\frac{1}{2}ab\sin\theta$, where a and b are the lengths of two of the sides and θ is the angle between these two sides, together with the *Cosine Rule*.



We label the vertices of the triangle as shown in the figure and let $\angle PQR = \theta$. It follows that $88 = \frac{1}{2}(22 \times 10)\sin\theta$. That is, $88 = 110\sin\theta$. Hence

$$\sin\theta = \frac{88}{110} = \frac{4}{5}.$$

From the identity $\cos^2\theta + \sin^2\theta = 1$, it follow that

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{4}{5}\right)^2 = 1 - \frac{16}{25} = \frac{9}{25} = \left(\frac{3}{5}\right)^2,$$

and therefore $\cos\theta = \pm\dfrac{3}{5}$.

From the diagram in the question, we see that $\angle PQR$ is an acute angle, that is, $0 < \theta < 90°$. Therefore $\cos\theta > 0$ and it follows that $\cos\theta = \frac{3}{5}$.

We can now deduce, using the Cosine Rule, that

$$\begin{aligned} y^2 &= PQ^2 + QR^2 - 2PQ.QR\cos\theta \\ &= 10^2 + 22^2 - 2 \times 10 \times 22\cos\theta \\ &= 100 + 484 - 2 \times 10 \times 22 \times \frac{3}{5} \\ &= 100 + 484 - 264 \\ &= 320. \end{aligned}$$

Hence $y = \sqrt{320} = 8\sqrt{5}$.

Commentary

In the problems below we cover a third method for answering Question 14. This method uses *Heron's Formula* for the area of a triangle.

In Method 1 we used a formula for the area of a triangle in terms of the height of the triangle. In Method 2 we used a formula which involves the angle between two sides. The advantage of Heron's Formula is that it gives the area of a triangle just in terms of the lengths of the sides of the triangle. However, as you will see if you tackle the problems below, this method involves some quite complicated algebra.

Heron's Formula: The area, A, of a triangle whose sides have lengths a, b and c is given by

$$A = \tfrac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

This formula may also be written as

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$, is half of the length of the perimeter of the triangle.

For investigation

14.1 Use Heron's Formula to find the third side, y, of an acute-angled triangle with sides 10 and 22, and area 88.

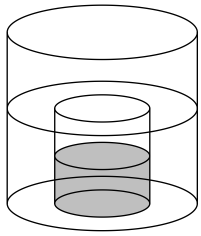
14.2 Deduce Heron's Formula from the formula $A = \frac{1}{2}ah$, where h is the height of the triangle. [Hint: in the figure for Method 1, let $QN = x$. Then use Pythagoras' Theorem, applied to triangles PNQ and PNR, to obtain an expression for x in terms of a, b and c.]

14.3 Deduce Heron's Formula from the formula $A = \frac{1}{2}ab\sin\theta$. [Hint: use the Cosine Formula to obtain an expression for $\cos\theta$ in terms of a, b and c.]

Note

Heron of Alexandria lived in the first century of the current era. He was a noted geometer and writer on mechanics. He invented many machines including a steam turbine. His proof of his formula for the area of a triangle is to be found in his book *Metrica*.

15. Two vases are cylindrical in shape. The larger vase has diameter 20 cm. The smaller vase has diameter 10 cm and height 16 cm. The larger vase is partially filled with water. Then the empty smaller vase, with the open end at the top, is slowly pushed down into the water, which flows over its rim. When the smaller vase is pushed right down, it is half full of water.



What was the original depth of the water in the larger vase?

A 10 cm B 12 cm C 14 cm D 16 cm E 18 cm

SOLUTION **C**

We use the formula $V = \pi r^2 h$ for the volume V of a cylinder of radius r and height h. Expressed in terms of the diameter d of the cylinder, where $r = \frac{1}{2}d$, this becomes $V = \frac{1}{4}\pi d^2 h$.

When the smaller vase is pushed right down and the water finishes flowing over the rim, the water in the larger vase comes to the top of the smaller cylinder and so has the same depth as height of the smaller cylinder, that is, 16 cm.

So the volume of water in the larger vase is that of a cylinder with diameter 20 cm and height 16 cm, less the volume of the empty half of the smaller vase.

The volume of a cylinder with diameter 20 cm and height 16 cm is, $\frac{1}{4}\pi \times 20^2 \times 16\,\text{cm}^3 = 1600\pi\,\text{cm}^3$. The volume of the empty half of the smaller cylinder is $\frac{1}{4}\pi \times 10^2 \times 8\,\text{cm}^3 = 200\pi\,\text{cm}^3$.

It follows that the volume of the water in the cylinder is, in cm^3,

$$1600\pi - 200\pi = 1400\pi.$$

Now suppose that the original depth of the water in the cylinder is x cm. It follows that the volume of the water in the cylinder is, in cm^3,

$$\tfrac{1}{4}\pi \times 20^2 \times x = 100\pi x.$$

Since these two expressions for the volume of the water must be the same,

$$100\pi x = 1400\pi,$$

from which it follows that $x = 14$.

Therefore the original depth of the water in the cylinder was 14 cm.

16. Fnargs are either red or blue and have 2, 3 or 4 heads. A group of six Fnargs consisting of one of each possible form is made to line up such that no immediate neighbours are the same colour nor have the same number of heads.

How many ways are there of lining them up from left to right?

A 12 B 24 C 60 D 120 E 720

Solution **A**

We let R2, R3, R4 be red Fnargs with 2, 3 and 4 heads, respectively, and B2, B3, B4 be blue Fnargs with 2, 3 and 4 heads, respectively. We need to count the number of ways of lining up R2, R3, R4, B2, B3, B4 so that no two Fnargs that are next to each other have the same colour or the same number of heads.

Suppose that a row of these six Fnargs begins with R2 at the left hand end. The second Fnarg in the row must be blue and have 3 or 4 heads. So the row either begins R2, B3 or R2, B4.

If the row begins R2, B3 the third Fnarg must be red and cannot have 3 heads. Since R2 is already in the line up, the third Fnarg must be R4. The fourth Fnarg must be blue and cannot have 4 heads. Since B3 is already in the row, this fourth Fnarg must be B2. This leaves just R3 and B4 to be placed. So, as the colours must alternate, the fifth and sixth Fnargs in the row must be R3 and B4 in this order from left to right. Hence the line up must be

R2, B3, R4, B2, R3, B4.

A similar argument shows that if the row begins R2, B4, then the line up must be as above but with 3 and 4 interchanged, that is,

R2, B4, R3, B2, R4, B3.

So there are just two ways to complete a row that begins with R2.

A similar argument shows that whichever Fnarg begins a row, there are just two ways to complete the row.

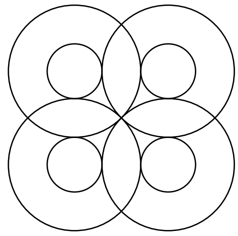
Since the row may begin with any of the six Fnargs, the total number of ways to line them up is $6 \times 2 = 12$.

For investigation

16.1 As a result of a genetic modification there are now Fnargs with 2, 3, 4 or 5 heads, but still only red or blue. In how many ways can we line up eight Fnargs consisting of one of each possible form so that two adjacent Fnargs have neither the same colour nor the same number of heads?

16.2 As a result of a further genetic modification there are now red, blue and green Fnargs, each with 2, 3, 4 or 5 heads. In how many ways can we line up twelve Fnargs consisting of one of each possible form so that two adjacent Fnargs have neither the same colour nor the same number of heads?

17. The diagram shows eight circles of two different sizes. The circles are arranged in concentric pairs so that the centres form a square. Each larger circle touches one other larger circle and two smaller circles. The larger circles have radius 1.

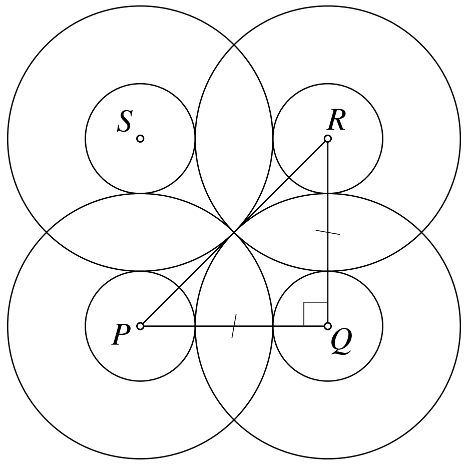


What is the radius of each smaller circle?

A $\frac{1}{3}$ B $\frac{2}{5}$ C $\sqrt{2} - 1$ D $\frac{1}{2}$ E $\frac{1}{2}\sqrt{2}$

Solution **C**

Let the centres of the circles be P, Q, R and S, as shown. Let the radius of the smaller circles be r.



Then the length of PQ is the sum of the radius of a larger circle and the radius of a smaller circle. That is, $PQ = 1 + r$. Similarly $QR = 1 + r$. Also, PR is the sum of the radii of two of the large circles. That is, $PR = 1 + 1 = 2$.

Because $PQRS$ is a square, $\angle PQR$ is a right angle.

Therefore, applying Pythagoras' theorem to the triangle PQR, we have

$$(1 + r)^2 + (1 + r)^2 = 2^2.$$

It follows that

$$2(1 + r)^2 = 4$$

and hence

$$(1 + r)^2 = 2.$$

Therefore

$$1 + r = \pm\sqrt{2}$$

and hence

$$r = \pm\sqrt{2} - 1.$$

Because a radius must be positive, we deduce that the radius of each smaller circle is

$$\sqrt{2} - 1.$$

.

18. What is the largest integer k whose square k^2 is a factor of $10!$?

$[10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1.]$

A 6 B 256 C 360 D 720 E 5040

Solution **D**

Commentary

The most straightforward way to answer this question would be to first evaluate $10!$, and then try out the squares of the given options, starting with the largest, until you find a square that is a factor of $10!$

If you do the multiplication you will find that $10! = 3\,628\,800$. If you now test the options, you will find that 5040^2 is not a factor of $10!$, but that 720^2 is.

However, this approach has several disadvantages. In the SMC calculators are not allowed, and this method requires a lot of arithmetic. Outside the context of SMC, we do not have only five options to test. We would have to start with the largest integer, k, such that $k^2 \leq 10!$, and check whether k^2 is a factor of $10!$. If not we would need to check whether $(k-1)^2$ is factor of $10!$, and so on, until we find the largest integer whose square is a factor of $10!$ Also, this method gives no insight into the problem and if 10 were replaced by a much larger number, it would not be feasible to use this method.

A better method is to work with the prime factorization of $10!$, using the fact that when we factorize a square into primes, each prime occurs to an even power. So we begin by factorizing $10!$ into primes, and then we look for the highest product of even powers of the prime factors that is a factor of $10!$

We express $10!$ as a product of prime numbers as follows.

$$\begin{aligned}
10! &= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\
&= (2 \times 5) \times 3^2 \times 2^3 \times 7 \times (2 \times 3) \times 5 \times 2^2 \times 3 \times 2 \\
&= 2^8 \times 3^4 \times 5^2 \times 7 \\
&= (2^4 \times 3^2 \times 5)^2 \times 7.
\end{aligned}$$

We deduce that the square of $2^4 \times 3^2 \times 5$ is a factor of $10!$ but there is no larger integer k such that k^2 is a factor of $10!$ So the largest integer k whose square k^2 is a factor of $10!$ is $2^4 \times 3^2 \times 5 = 16 \times 9 \times 5 = 720$.

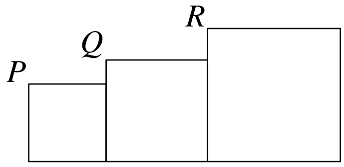
For investigation

18.1 Which is the largest integer k whose square, k^2, is a factor of $11!$?

18.2 Which is the largest integer k whose square, k^2, is a factor of $12!$?

18.3 Which is the least integer, n, such that 126^2 is a factor of $n!$?

19. Three squares are arranged as shown so that their bases lie on a straight line. Also, the corners P, Q and R lie on a straight line. The middle square has sides that are 8 cm longer than the sides of the smallest square. The largest square has sides of length 50 cm. There are two possible values for the length (in cm) of the sides of the smallest square.

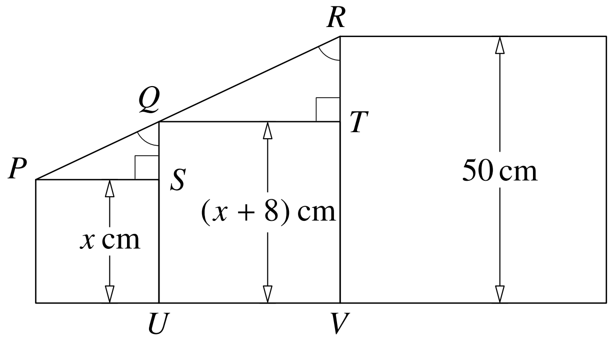


Which of the following are they?

A 2, 32 B 4, 42 C 4, 34 D 32, 40 E 34, 42

Solution **A**

We let S, T, U and V be the points shown in the figure.



We let the side length of the smallest square be x cm.

It follows that the middle square has sides of length $(x + 8)$ cm

Since the lines QU and RV are both perpendicular to UV they are parallel lines.

Therefore, as PQR is a straight line, the corresponding angles $\angle PQS$ and $\angle QRT$ are equal.

Because the figure is made up of squares, $\angle PSU = \angle QTV = 90°$. Therefore, as angles on a line add up to 180°, we have $\angle PSQ = \angle QTR = 90°$. It follows that the right-angled triangles PSQ and QTR are similar because they have the same angles. Therefore

$$\frac{QS}{PS} = \frac{RT}{QT}.$$

Now QS has length 8 cm, PS has length x cm, RT has length $(50 - (x + 8))$ cm $= (42 - x)$ cm and QT has length $(x + 8)$ cm. It follows that

$$\frac{8}{x} = \frac{42 - x}{x + 8}.$$

Multiplying both sides of this equation by $x(x + 8)$, we see that this last equation is equivalent to

$$8(x + 8) = x(42 - x),$$

that is,

$$8x + 64 = 42x - x^2.$$

This last equation may be rearranged to give

$$x^2 - 34x + 64 = 0.$$

We can now factorize the quadratic in this equation to give

$$(x - 32)(x - 2) = 0.$$

Therefore

$$x = 2 \text{ or } x = 32.$$

We deduce that the possible lengths, in cm, of the sides of the smallest square are 2 and 32.

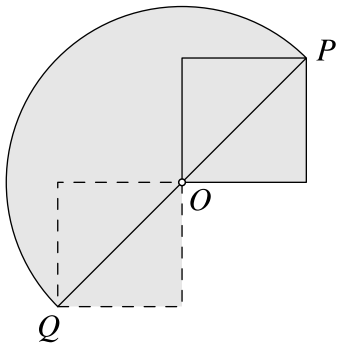
20. A square ink pad has sides of length 1 cm. It is covered in black ink and carefully placed in the middle of a piece of white paper. The square pad is then rotated 180° about one of its corners so that all of the pad remains in contact with the paper throughout the turn. The pad is then removed from the paper.

What area of paper, in cm^2, is coloured black?

A $\pi + 2$ B $2\pi - 1$ C 4 D $2\pi - 2$ E $\pi + 1$

Solution **E**

We let O be the corner of the square about which it rotates. The initial position of the square is shown by solid lines, and its final position by broken lines, so that the corner initially at the position P rotates to the position Q.



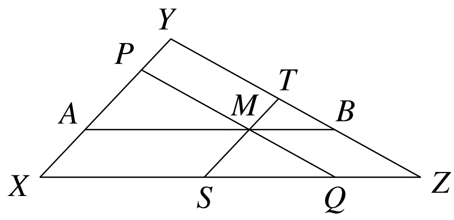
The area that is in contact with the paper at some time during the rotation is shown shaded. This is the area that gets coloured black as the square rotates. It is made up of the semicircle with diameter PQ and the two shaded half squares outside this semicircle.

The length of the sides of the squares, in cm, is 1. We let the length of OP be x cm. Then, by Pythagoras' Theorem, $x^2 = 1^2 + 1^2 = 2$. Therefore, the area of the semicircle with diameter PQ is, in cm^2, $\frac{1}{2}\pi x^2 = \frac{1}{2}\pi \times 2 = \pi$.

The area of the two half squares outside the semicircle is equal to the area of one square with side length 1 cm, that is, $1\,\text{cm}^2$.

Therefore, the total area that is coloured black is, in cm^2, $\pi + 1$.

21. The diagram shows a triangle XYZ. The sides XY, YZ and XZ have lengths 2, 3 and 4 respectively. The lines AMB, PMQ and SMT are drawn parallel to the sides of triangle XYZ so that AP, QS and BT are of equal length.



What is the length of AP?

A $\frac{10}{11}$ B $\frac{11}{12}$ C $\frac{12}{13}$ D $\frac{13}{14}$ E $\frac{14}{15}$

SOLUTION **C**

Let the common length of AP, QS and BT be x.

Because AB is parallel to XZ, the corresponding angles, $\angle PAM$ and $\angle YXZ$ are equal. Because PQ is parallel to YZ, the corresponding angles, $\angle APM$ and $\angle XYZ$ are equal.

It follows that the triangles APM and XYZ are similar. Therefore their corresponding sides are in the same ratio. Hence, in particular, $\frac{AP}{XY} = \frac{AM}{XZ}$. That is, $\frac{x}{2} = \frac{AM}{4}$, and therefore $AM = 2x$.

Since AM is parallel to XS, and XA is parallel to SM, the quadrilateral $MAXS$ is a parallelogram. Therefore $XS = AM = 2x$.

Similarly, the triangles MTB and XYZ are similar, and therefore $\frac{TB}{YZ} = \frac{MB}{XZ}$. Hence $\frac{x}{3} = \frac{MB}{4}$. Therefore $MB = \frac{4}{3}x$. Also $MBZQ$ is a parallelogram. Therefore $QZ = MP = \frac{4}{3}x$.

Now, as $XS + SQ + QZ = XZ$, it follows that

$$2x + x + \frac{4}{3}x = 4.$$

That is,

$$\frac{13}{3}x = 4,$$

from which it follows that

$$x = \frac{3}{13} \times 4 = \frac{12}{13}.$$

Therefore the length of AP is $\frac{12}{13}$.

FOR INVESTIGATION

21.1 Find the lengths of XA and PY in terms of x. Then use the fact that XY has length 2 to determine the value of x.

21.2 Similarly, determine the value of x by finding the lengths of YT and BZ, and then use the fact that YZ has length 3.

21.3 Is it true that for every triangle XYZ there is a point M, such that when the lines AMB, PMQ and SMT are drawn parallel to the sides of the triangle, then AP, QS and BT are of equal length ?

22. Let $f(x) = x + \sqrt{x^2+1} + \dfrac{1}{x - \sqrt{x^2+1}}$.

What is the value of $f(2015)$?

A -1 B 0 C 1 D $\sqrt{2016}$ E 2015

Solution **B**

Commentary

At first sight, this looks an impossibly difficult question, as it seems to involve working out the value of the square root $\sqrt{x^2+1}$ for $x = 2015$, *without the use of a calculator*! As this cannot be the intended method, we look for an alternative approach.

The presence of both the terms $x - \sqrt{x^2+1}$ and $x + \sqrt{x^2+1}$ in the expression for $f(x)$ suggests that we can make progress using algebra, and in particular, the *difference of two squares* formula $(a-b)(a+b) = a^2 - b^2$, with $a = x$, and $b = \sqrt{x^2+1}$.

Indeed, if you have the confidence to try this approach, you will see that this question turns out not to be at all difficult.

We have

$$f(x) = x + \sqrt{x^2+1} + \frac{1}{x - \sqrt{x^2+1}}.$$

We now put the two terms involved in $f(x)$ over a common denominator. This gives

$$\begin{aligned} f(x) &= \frac{\left(x - \sqrt{x^2+1}\right)\left(x + \sqrt{x^2+1}\right) + 1}{x - \sqrt{x^2+1}} \\ &= \frac{\left(x^2 - (\sqrt{x^2+1})^2\right) + 1}{x - \sqrt{x^2+1}} \\ &= \frac{\left(x^2 - (x^2+1)\right) + 1}{x - \sqrt{x^2+1}} \\ &= \frac{-1+1}{x - \sqrt{x^2+1}} \\ &= \frac{0}{x - \sqrt{x^2+1}} \\ &= 0. \end{aligned}$$

This holds whatever the value of x. Therefore, in particular, $f(2015) = 0$.

23. Given four different non-zero digits, it is possible to form 24 different four-digit numbers containing each of these four digits.

What is the largest prime factor of the sum of the 24 numbers?

A 23 B 93 C 97 D 101 E 113

SOLUTION **D**

Let the four different non-zero digits be a, b, c and d. If we use them to make 24 different four-digit numbers, then each of a, b, c and d occurs 6 times in the units position, 6 times in the tens position, 6 times in the hundreds position, and 6 times in the thousands position.

It follows that the digits in the units position contribute $6(a + b + c + d)$ to the sum of all 24 numbers, and the digits in the tens position contribute $6(a + b + c + d) \times 10 = 60(a + b + c + d)$ to this sum. Similarly, the digits in the hundreds position contribute $600(a + b + c + d)$ and those in the thousands column contribute $6000(a + b + c + d)$.

Therefore the total sum of all 24 numbers is

$$6000(a+b+c+d)+600(a+b+c+d)+60(a+b+c+d)+6(a+b+c+d) = 6666(a+b+c+d).$$

We can factorize 6666 into primes, as follows,

$$6666 = 6 \times 1111 = 2 \times 3 \times 11 \times 101.$$

Since a, b, c, d are four different digits, their sum is at most $9 + 8 + 7 + 6 = 30$ and so this sum cannot have a prime factor as large as 101. We deduce that the largest prime factor of the sum, $6666(a + b + c + d)$, of all the 24 numbers is 101.

FOR INVESTIGATION

23.1 Given five different non-zero digits, it is possible to form 120 different five-digit numbers containing each of these five digits.

What is the largest prime factor of the sum of the 120 numbers?

23.2 Given six different non-zero digits, how many different six-digit numbers are there which contain each of the given six digits?

What is the largest prime factor of the sum of all these six-digit numbers?

24. Peter has 25 cards, each printed with a different integer from 1 to 25. He wishes to place N cards in a single row so that the numbers on every adjacent pair of cards have a prime factor in common.

What is the largest value of N for which this is possible?

A 16 B 18 C 20 D 22 E 24

SOLUTION **C**

An integer can occur on a card in the row only if it shares a prime factor with at least one other integer on a card in the row. This rules out 1, which has no prime factors, and the primes 13, 17, 19 and 23 which are not factors of any other integers in the range from 1 to 25.

With these five integers excluded, this leaves at most 20 cards that can be in the row. It is possible to arrange all these remaining cards in a row so as to meet the condition that integers on adjacent cards share a prime factor. In fact, there are lots of ways to do this. For example

$$7, 14, 21, 3, 18, 15, 5, 25, 10, 4, 20, 24, 9, 6, 8, 12, 16, 2, 22, 11.$$

It follows that the largest value of N with the required property is 20.

FOR INVESTIGATION

24.1 There are lots of different ways to arrange the cards carrying the 20 integers

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25$$

in a row so that the integers on adjacent cards have a prime factor in common. Just one of these arrangements is given in the above solution.

One of these integers must occur at the end, either first or last, in any row of all these integers that meet the given condition. Which is it?

24.2 For which of the integers, n, of the 20 integers listed in Exercise 24.1, is it possible to arrange all 20 cards in a row so that n is on the first card of the row, and so that every pair of adjacent integers in the row have a prime factor in common?

24.3 As we say above, there are lots of ways to arrange the 20 cards in a row so that every pair of adjacent integers has a prime number in common. Counting all the different possible solutions is rather hard. You can try this if you like, but we suggest that instead you consider the case where there are just 12 cards, each printed with a different integer from 1 to 12.

(i) What is the largest value of N such that N of the 12 cards can be placed in a single row so that the numbers on every adjacent pair of cards have a prime factor in common?

(ii) For this value of N how many different ways are there to arrange the N cards in a row so as to meet the required condition?

25. A function, defined on the set of positive integers, is such that $f(xy) = f(x) + f(y)$ for all x and y. It is known that $f(10) = 14$ and $f(40) = 20$.

What is the value of $f(500)$?

A 29 B 30 C 39 D 48 E 50

SOLUTION **C**

COMMENTARY

The factorization of 500 into primes is $500 = 2^2 \times 5^3$. Since the given function satisfies the formula $f(xy) = f(x) + f(y)$, it follows that $f(500) = f(2 \times 2 \times 5 \times 5 \times 5) = f(2) + f(2) + f(5) + f(5) + f(5) = 2f(2) + 3f(5)$.

Therefore we can find the value of $f(500)$ if we can find the values of $f(2)$ and $f(5)$. In the first method given below we go about this directly. The second method is more systematic, but more complicated.

METHOD 1

Using the formula $f(xy) = f(x) + f(y)$ with $x = 4$ and $y = 10$, gives $f(40) = f(4) + f(10)$. Hence $f(4) = f(40) - f(10) = 20 - 14 = 6$. Using the formula with $x = y = 2$, gives $f(4) = f(2) + f(2)$. Therefore $f(2) = \frac{1}{2}f(4) = \frac{1}{2} \times 6 = 3$. Using the formula with $x = 2$ and $y = 5$, gives $f(10) = f(2) + f(5)$ and therefore $f(5) = f(10) - f(2) = 14 - 3 = 11$.

Since $500 = 2^2 \times 5^3$, we can now deduce that $f(500) = 2f(2) + 3f(5) = 2 \times 3 + 3 \times 11 = 6 + 33 = 39$.

METHOD 2

Let $f(2) = a$ and $f(5) = b$. Since $10 = 2 \times 5$, we have $f(10) = f(2) + f(5)$. Since $40 = 2^3 \times 5$, we have $f(40) = 3f(2) + f(5)$.

Therefore, as $f(10) = 14$ and $f(40) = 20$, we obtain the two linear equations

$$a + b = 14 \quad (1)$$
$$3a + b = 20 \quad (2)$$

From equations (1) and (2)

$$(3a + b) - (a + b) = 20 - 14.$$

We deduce that $2a = 6$ and therefore $a = 3$. Hence, by equation (1), we have $b = 11$.

Since $500 = 2^2 \times 5^3$, we deduce that $f(500) = 2f(2) + 3f(5) = 2 \times 3 + 3 \times 11 = 39$.

FOR INVESTIGATION

25.1 Assume that $f(xy) = f(x) + f(y)$ for all positive integers x and y. What is the value of $f(1)$?

25.2 Again, assume that $f(xy) = f(x) + f(y)$ for all positive integers x and y. Show that if the positive integer n has the factorization $n = p^a \times q^b \times r^c$, then $f(n) = af(p) + bf(q) + cf(r)$.